

The String Uncertainty Relations follow from the New Relativity Principle

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Abstract

The String Uncertainty Relations have been known for some time as the stringy corrections to the original Heisenberg's Uncertainty principle. In this letter the Stringy Uncertainty relations, and corrections thereof, are explicitly derived from the New Relativity Principle that treats all dimensions and signatures on the same footing and which is based on the postulate that the Planck scale is the minimal length in Nature in the same vein that the speed of light was taken as the maximum velocity in Einstein's theory of Special Relativity. The Regge behaviour of the string's spectrum is also a natural consequence of this New Relativity Principle.

Recently we have proposed that a New Relativity principle may be operating in Nature which could reveal important clues to find the origins of M theory [1]. We were forced to introduce this new Relativity principle, where all dimensions and signatures of spacetime are on the same footing, to find a fully covariant formulation of the p -brane Quantum Mechanical Loop Wave equations. This New Relativity Principle, or the principle of Polydimensional Covariance as has been called by Pezzaglia, has also been crucial in the derivation of Papapetrou's equations of motion of a spinning particle in curved spaces that was a long standing problem which lasted almost 50 years [2]. A Clifford calculus was used where all the equations were written in terms of Clifford-valued multivector quantities; i.e one had to abandon the use of vectors and tensors and replace them by Clifford-algebra valued quantities, matrices, for example .

In this letter we will explicitly derive the String Uncertainty Relations, and corrections thereof, directly from the Quantum Mechanical Wave equations on Noncommutative Clifford manifolds or **C**-spaces [1]. There was a one-to-one correspondence between the nested hierarchy of point, loop, **2**-loop, **3**-loop,.....**p**-loop histories encoded in terms of hypermatrices and wave equations written in terms of Clifford-algebra valued multivector quantities. This permits us to recast the QM wave equations associated with the hierarchy of nested **p**-loop histories, embedded in a target spacetime of D dimensions , where the values of p range from : $p = 3D0, 1, 2, 3, \dots, D-1$, as a *single* QM line functional wave equation whose lines live in a Noncommutative Clifford manifold of 2^D dimensions. $p = 3DD-1$ is the the maximum value of p that saturates the embedding spacetime dimension.

The line functional wave equation in the Clifford manifold, **C**-space is :

$$\int d\Sigma \left(\frac{\delta^2}{\delta X(\Sigma) \delta X(\Sigma)} + \mathcal{E}^2 \right) \Psi[X(\Sigma)] = 3D0. \quad (1)$$

where Σ is an invariant evolution parameter of l^D dimensions

generalizing the notion of the invariant proper time in Special Relativity linked to a massive point particle line (path) history :

$$(d\Sigma)^2 = 3D(d\Omega_{p+1})^2 + \Lambda^{2p}(dx^\mu dx_\mu) + \Lambda^{2(p-1)}(d\sigma^{\mu\nu} d\sigma_{\mu\nu}) + \Lambda^{2(p-2)}(d\sigma^{\mu\nu\rho} d\sigma_{\mu\nu\rho}) + \dots \quad (2)$$

Λ is the Planck scale in D dimensions. $\mathbf{X}(\Sigma)$ is a Clifford-algebra valued " line " living in the Clifford manifold (**C**-space) :

$$X = 3D\Omega_{p+1} + x_\mu \gamma^\mu + \sigma_{\mu\nu} \gamma^\mu \gamma^\nu + \sigma_{\mu\nu\rho} \gamma^\mu \gamma^\nu \gamma^\rho + \dots \quad (3a)$$

The multivector \mathbf{X} encodes in one single stroke the point history represented by the ordinary x_μ coordinates and the holographic projections of the nested family of **1**-loop, bf **2**-loop, **3**-loop...**p**-loop histories onto the embedding coordinate spacetime planes

$$\sigma_{\mu\nu}, \sigma_{\mu\nu\rho}, \dots, \sigma_{\mu_1 \mu_2 \dots \mu_{p+1}} \quad (3b)$$

respectively. The scalar Ω_{p+1} is the invariant proper $p+1 = 3DD$ volume associated with the motion of the (maximal dimension) \mathbf{p} -loop across the $D = 3Dp + 1$ -dim target spacetime. There was a coincidence condition [1] that required to equate the values of the center of mass coordinates x_μ , for all the \mathbf{p} -loops, with the values of the x^μ coordinates of the point particle path history. This was due to the fact that upon setting $\Lambda = 3D0$ all the \mathbf{p} -loop histories collapse to a point history. The latter history is the baseline where one constructs the whole hierarchy. This also required a proportionality relationship :

$$\tau \sim \frac{A}{\Lambda} \sim \frac{V}{\Lambda^2} \sim \dots \sim \frac{\Omega^{p+1}}{\Lambda^p}. \quad (4)$$

$\tau, A, V, \dots, \Omega^{p+1}$ represent the invariant proper time, proper area, proper volume, ... proper $p+1$ -dim volume associated with a point, loop, **2**-loop, **3**-loop, ... \mathbf{p} -loop histories, respectively. $\mathcal{E} = 3DT$ is a quantity of dimension $mass^{p+1}$, the maximal p -brane tension ($p = 3DD - 1$).

The wave functional Ψ is in general a Clifford-valued, hypercomplex number. In particular it could be a complex, quaternionic or octonionic valued quantity. At the moment we shall not dwell on the very subtle complications and battles associated with the quaternionic/octonionic extensions of Quantum Mechanics [14] based on Division algebras and simply take the wave function to be a complex number. The line functional wave equation for lines living in the Clifford manifold (\mathbf{C} -spaces) are difficult to solve in general. To obtain the String Uncertainty Relations, and corrections thereof, one needs to simplify them. The most simple expression is to write the wave equation in units $c = 3D1$:

$$[-\hbar^2(\frac{\partial^2}{\partial x^\mu \partial x_\mu} + \frac{\Lambda^2}{2} \frac{\partial^2}{\partial \sigma^{\mu\nu} \partial \sigma_{\mu\nu}} + \frac{\Lambda^4}{3!} \frac{\partial^2}{\partial \sigma^{\mu\nu\rho} \partial \sigma_{\mu\nu\rho}} + \dots) + \Lambda^{2p} \mathcal{E}^2] \Psi[x^\mu, \sigma^{\mu\nu}, \sigma^{\mu\nu\rho}, \dots] = 3D0 \quad (5)$$

where we have dropped the first component of the Clifford multivector dependence, Ω^{p+1} , of the wave functional and we have replaced functional differential equations for ordinary differential equations. Similar types of equations in a different context (Papapetrou's equations) with only the first two terms of eq-(5), have also been written in [2].

The last equation contains the seeds of the String Uncertainty Relations and corrections thereof. Plane wave type solutions to eq-(5) are :

$$\Psi = 3De^{i(k_\mu x^\mu + (k_\mu \wedge k_\nu) \sigma^{\mu\nu} + \dots)}. \quad (6)$$

inserting the plane wave solution into the simplified wave equation yields the dispersion relation :

$$\hbar^2(k^2 + \frac{1}{2}\Lambda^2(k_\mu \wedge k_\nu)(k^\mu \wedge k^\nu) + \frac{1}{3!}\Lambda^4(k_\mu \wedge k_\nu \wedge k_\rho)(k^\mu \wedge k^\nu \wedge k^\rho) + \dots) + \Lambda^{2p} \mathcal{E}^2 = 3D0. \quad (7)$$

this is just the generalization of the ordinary wave/particle dispersion relationship

$$p^2 = 3D\hbar^2 k^2. \quad p^2 + m^2 = 3D0. \quad (8)$$

On dimensional grounds one has that :

$$(k_\mu \wedge k_\nu)(k^\mu \wedge k^\nu) \sim (k^2)^2 = 3Dk^4. (k_\mu \wedge k_\nu \wedge k_\rho)(k^\mu \wedge k^\nu \wedge k^\rho) \sim (k^3)^2 = 3Dk^6 \dots \dots \dots \quad (9)$$

where the proportionality factors in eq-(9) are just contractions of two

$\epsilon^{123\dots D}$ tensors in D dimensions. Factoring out the k^2 factor in (7), using the analog of the dispersion relation (8) and taking the square root, after performing the binomial/Taylor expansion of the square root, subject to the condition $\Lambda^2 k^2 < 1$, one obtains an *effective* energy dependent Planck " constant " that takes into account the Noncommutative nature of the Clifford manifold (\mathbf{C} -space) at Planck scales :

$$\hbar_{eff}(k^2) = 3D\hbar(1 + \frac{1}{2.2!}\beta_1 \Lambda^2 k^2 + \frac{1}{2.3!}\beta_2 \Lambda^4 k^4 + \dots \dots \dots). \quad (10)$$

where we have included explicitly the D dependent coefficients β_1, β_2, \dots that arise as a result of the contraction of the $\epsilon^{123\dots D}$ tensors in (9).

Arguments concerning an effective value of Planck's " constant " related to higher derivative theories and the modified uncertainty relations have been given by [8]. The advantage of this derivation based on the New Relativity Principle is that one automatically avoids the problems involving the ad hoc introduction of higher derivatives in Physics (ghosts, ...) .

The uncertainty relations for the coordinates-momenta follow from the Heisenberg-Weyl algebraic relation familiar in QM :

$$\Delta x \Delta p \geq | \langle [\hat{x}, \hat{p}] \rangle |. \quad [\hat{x}, \hat{p}] = 3Di\hbar \quad (11)$$

Now we have that in \mathbf{C} -spaces, x, p must not, and should not, be interpreted as ordinary vectors of spacetime but as one of the many *components* of the Clifford-algebra valued multivectors that " coordinatize " the Noncommutative Clifford Manifold, \mathbf{C} -space. The Noncommutativity is *encoded* in the *effective* value of the Planck's " constant " which *modifies* the Heisenberg-Weyl x, p algebraic commutation relations and, consequently, generates new uncertainty relations :

$$\Delta x \Delta p \geq | \langle [x, p] \rangle | = 3D \langle \hbar_{eff} \rangle = 3D\hbar(1 + \frac{1}{2.2!}\beta_1\Lambda^2 \langle k^2 \rangle + \frac{1}{2.3!}\beta_2\Lambda^4 \langle k^4 \rangle + \dots) \quad (12)$$

Using the relations :

$$\hbar k = 3Dp. \quad \langle p^2 \rangle \geq (\Delta p)^2. \quad \langle p^4 \rangle \geq (\Delta p)^4 \dots \quad (13)$$

one arrives at :

$$\Delta x \Delta p \geq \hbar + \frac{\beta_1\Lambda^2}{4\hbar}(\Delta p)^2 + \frac{\beta_2\Lambda^4}{12\hbar^3}(\Delta p)^4 + \dots \quad (14)$$

Finally, keeping the first two terms in the expansion in the r.h.s of eq- (14) one recovers the ordinary String Uncertainty Relation [5] directly from the= New Relativity Principle as promised :

$$\Delta x \geq \frac{\hbar}{\Delta p} + \frac{\beta_1\Lambda^2}{4\hbar}(\Delta p). \quad (15)$$

which is just a reflection of the minimum distance condition in Nature [3,4,5,6,7,10] and an inherent Non-commutative nature of the Clifford manifold (\mathbf{C} -space) .

There is a widespread misunderstanding about the modification of the Heisenberg-Weyl algebra (12). One could start from a canonical pair of variables q, p and perform a *noncanonical* change of variables Q, P such as to precisely reproduce the modified commutation relations of eq-(12) :

$$x \rightarrow x'. \quad p \rightarrow p' = 3Df(p). \quad [x', p'] = 3D[x, f(p)] = 3Di\hbar[\frac{\partial}{\partial p}, f(p)] = 3Di\hbar\frac{\partial f(p)}{\partial p} = 3Dr.h.s \text{ of eq- (12)}. \quad (16)$$

This *noncanonical* change of coordinates is *not* what is represented here by the modified Heisenberg-Weyl algebra. Space at small scales is *not* necessarily governed by the familiar Lorentzian symmetries : it is a Noncommutative Clifford manifold that requires abandoning the naive notion of vectors and tensors and replacing them by Clifford multivectors. In other words, it is a world where Quantum Groups operate . The fact that the String Uncertainty relations reflect the existence of a minimum length in Nature is consistent with the *discretization* of spacetime at the Planck scale [9] and the replacement of ordinary Lorentzian group symmetries by Quantum Group (Hopf Algebras) Symmetries [12].

The New Relativity Principle reshuffles, for example, a loop history into a membrane history; a membrane history into a 5-brane history and so forth; " Lorentz " transformations in \mathbf{C} -spaces involve hypermatrix changes of " coordinates " [1] . The naive Lorentz transformations do not apply in the world

of Planck scale physics. Only at large scales the Riemannian continuum is recaptured . For a discussion of the more fundamental Finsler Geometries

implementing the minimum scale (maximal proper acceleration) in String Theory see [13].

The New Relativity principle not only reproduces the ordinary String Uncertainty Relations but yields *corrections* thereof in one single stroke as we have shown in eq-(14)! This is a positive sign that the New Relativity principle is on the right track to reveal the geometrical foundations of *M* theory . Uncertainty relations based on the Scale Relativity theory [3] were furnished in [10]. We must emphasize that the latter uncertainty relations involved spacetime *resolutions*. Resolutions are *not* statistical uncertainties, therefore the relations [10] *cannot* be used to evaluate the modified coordinates-momenta commutation relations like the r.h.s of (12).

To finalize this letter we will show how the Regge trajectories behaviour of the string's spectrum emerges also from the New Relativity principle. Pezzaglia's derivation of Papapetrou's equations [2] for a spinning particle moving in curved spaces were based on an invariant interval of the form :

$$(d\Sigma)^2 = 3Ddx^\mu dx_\mu + \frac{1}{2\lambda^2} d\sigma^{\mu\nu} d\sigma_{\mu\nu}. \quad (17a)$$

where λ is a length scale. The norm of the Clifford-valued momentum is :

$$P^2 = 3Dp_\mu p^\mu - \frac{1}{2\lambda^2} S_{\mu\nu} S^{\mu\nu}. \quad (17b)$$

where $S^{\mu\nu}$ is the spin or canonically-conjugate variable to the *area* . If we set the $\lambda \sim \Lambda$ and relate the squared-norm $\|S^{\mu\nu}\|^2$ to the value of the norm-squared of the **2**-vector conjugate to the holographic area variables $\sigma^{\mu\nu}$; i.e $(k^\mu \wedge k^\nu)(k_\mu \wedge k_\nu)$, norm-squared which is proportional to k^4 , one can infer, after inserting the appropriate units ($c = 3D1$), from the spin-squared terms of eq-(17b) and the dispersion relation given by eq-(7) :

$$\frac{\|S^{\mu\nu}\|}{\hbar} \sim \Lambda^2 k^2 = 3D \frac{\Lambda^2 p^2}{\hbar^2} = 3D \frac{\Lambda^2 m^2}{\hbar^2} = 3Dn \frac{\Lambda^2 m_P^2}{\hbar^2} = 3Dn. \quad (18)$$

hence one has that the spin is quantized in units of \hbar and from the third term in the r.h.s of eq-(18) one recovers the Regge trajectories behaviour of the string spectrum in units where $\hbar = 3Dc = 3D1$:

$$J \sim \alpha' m^2 + a. \quad m^2 \sim nm_P^2. \quad \alpha' \sim \Lambda^2 = 3D \frac{1}{m_P^2}. \quad (19)$$

which is consistent with the action-angle variables relation :

$$S = 3D \int P_{\mu\nu} d\sigma^{\mu\nu} \sim n\hbar. \quad (20)$$

where $P_{\mu\nu}$ is the area-momentum variable conjugate to the string areal interval $d\sigma^{\mu\nu}$. The action is a multiple of \hbar as it should.

Having derived the string uncertainty relations, and corrections thereof, and explained in simple terms why the Regge trajectories occur in the string spectrum should be enough encouragement to proceed forward with the New Relativity Principle.

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References

1. C. Castro : " Hints of a New Relativity Principle from *p*-brane Quantum Mechanics " hep-th/9912113.
2. W. Pezzaglia : " Dimensionally Democratic Calculus and Principles of Polydimensional Physics " gr-qc/9912025.
3. L. Nottale : Fractal Spacetime and Microphysics, Towards the Theory of Scale Relativity

- World Scientific 1992.
- L. Nottale : La Relativite dans Tous ses Etats. Hachette Literature. Paris. 1999.
4. M. El Naschie : Jour. Chaos, Solitons and Fractals **vol 10** nos. 2-3 (1999) 567.
5. D. Amati, M. Ciafaloni, G. Veneziano : Phys. Letts **B 197** (1987) 81.
- D. Gross, P. Mende : Phys. Letts **B 197** (1987) 129.
6. L. Garay : Int. Jour. Mod. Phys. **A 10** (1995) 145.
7. A. Kempf, G. Mangano : " Minimal Length Uncertainty and Ultraviolet Regularization " hep-th/9612084.
- G. Amelino-Camelia, J. Lukierski, A. Nowicki : " κ deformed covariant phase space and Quantum Gravity Uncertainty Relations " hep-th/9706031.
8. R. Adler, D. Santiago : " On a generalization of Quantum Theory : Is the Planck Constant Really Constant ? " hep-th/9908073
9. C. Rovelli : " The century of the incomplete revolution...." hep-th/9910131
10. C. Castro : Foundations of Physics Letts **10** (1997) 273.
11. A. Connes : Noncommutative Geometry. Academic Press. New York. 1994.
12. S. Mahid : Foundations of Quantum Group Theory. Cambridge University Press. 1995. Int. Jour. Mod. Phys **A 5** (1990) 4689.
- L.C. Biedenharn, M. A. Lohe : Quantum Groups and q-Tensor Algebras . World Scientific. Singapore . 1995.
13. H. Brandt : Jour. Chaos, Solitons and Fractals **10** nos 2-3 (1999) 267.
14. S. Adler : Quaternionic Quantum Mechanics and Quantum Fields . Oxford, New York. 1995.